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Number of spiral self-avoiding loops on a triangular lattice

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Abstract. Exact results are derived for the number C_n of spiral self-avoiding loops with n steps on a triangular lattice. A closed-form expression for the generating function

$$G(x)=\sum_{n=1}^{\infty} C_n x^n$$

is obtained. For $n\to\infty$, C_n increases as $n^6/31\,104$.

Spiral self-avoiding walks (SSAW) were first considered by Privman (1983). Exact results for the number of n-step SSAW on a square lattice were derived by Blöte and Hilhorst (1984), Guttmann and Wormald (1984) and Joyce (1984). Recently the number C_n of n-step spiral self-avoiding loops on a square lattice was calculated exactly by Manna (1985). His result is

$$C_n=(n-2)(n^2-4n+24)/48 \tag{1}$$

where n is an even integer. For $n\to\infty$, C_n increases as $n^3/48$.

SSAW on a triangular lattice are those self-avoiding walks which, at each step, either go straight ahead, turn through 60° or turn through 120° (both to the left). The special case where 60° deviations are forbidden has been solved exactly by several authors (Lin 1985, Joyce and Brak 1985, Liu and Lin 1985, Lin and Liu 1986).

Our aim in the present paper is to study the asymptotic behaviour of spiral self-avoiding loops on a triangular lattice and to derive an exact expression for the generating function

$$G(x)=\sum_{n=1}^{\infty} C_n x^n. \tag{2}$$

Consider first the special case where 60° deviations are forbidden. The spiral constraint restricts the loops to be of two categories, triangular and non-triangular types, as shown in figure 1. The starting direction is fixed to the right. We define $n=3k$ where k is an integer. A triangular loop with perimeter length n will occur k times, depending on the position of the starting point on the horizontal side. Each non-triangular loop corresponds one-to-one with an internal lattice point of a triangle with perimeter n. Therefore we have

$$C_n=k+(k-1)(k-2)/2=(k^2-k+2)/2. \tag{3}$$

The generating function is

$$\begin{aligned} G(x)&=\sum_{k=1}^{\infty} \tfrac{1}{2}(k^2-k+2)x^{3k} \\ &=x^3(1-x^3+x^6)(1-x^3)^{-3}. \end{aligned} \tag{4}$$

For $n\to\infty$, C_n increases as $n^2/18$.

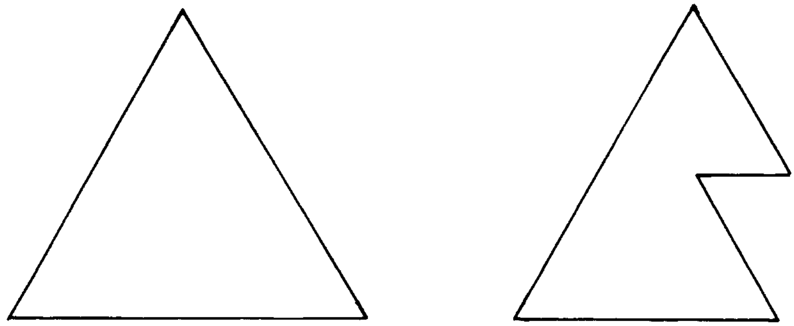

Figure 1. Two types of spiral self-avoiding loops where the 60° deviations are forbidden.

Consider next the special case where 120° deviations are forbidden. The constraint restricts the loops to be of four categories, as shown in figure 2. The generating function corresponding to figure $2(k)$ is denoted by G_k. We have

$$
\begin{aligned}
G_1 &= \sum_{a,b,c=1}^{\infty} a x^{3a+2b+2c} \sum_{\min(a+c,a+b)>d>0} x^{-d} \\
G_2 &= \sum_{a,b,c=1}^{\infty} (a-1)(c-1) x^{3a+2b+2c} \sum_{\min(a+c,a+b)>d>0} x^{-d} \\
G_3 &= \sum_{a,b,c=1}^{\infty} x^{3a+2b+2c}
\end{aligned}
\tag{5}
$$

We find

$$
\begin{aligned}
G_1+G_2 &= \sum C'_n x^n \\
&= x^6(1-x^2)^{-4}(1-x^3)^{-2}(1+x^3+x^4+x^5+x^8) \\
&= x^6+4x^8+3x^9+11x^{10}+13x^{11}+29x^{12}+36x^{13}+68x^{14}+85x^{15}+\ldots \\
G_3 &= \sum C''_n x^n \\
&= x^7(1-x^2)^{-2}(1-x^3)^{-1} \\
&= x^7+2x^9+x^{10}+3x^{11}+2x^{12}+5x^{13}+3x^{14}+7x^{15}+\ldots
\end{aligned}
\tag{6}
$$

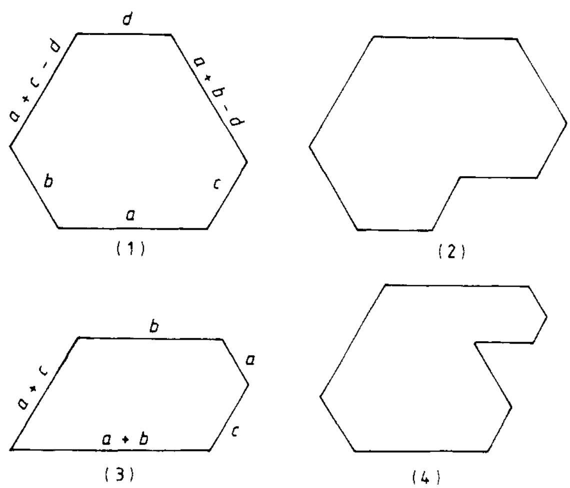

Figure 2. Four types of spiral self-avoiding loops where the 120° deviations are forbidden.

where $C'_n \to n^5/3456$ and $C''_n \to n^2/24$ when $n \to \infty$. The calculation of G_4 is very complicated. After a lengthy derivation, we obtain

$$\begin{aligned} G_4 &= \sum C'''_n x^n \\ &= x^{11}(1-x^2)^{-4}(1-x^3)^{-3}(1+x+3x^2+2x^3+2x^4+x^6) \\ &= x^{11}+x^{12}+7x^{13}+9x^{14}+27x^{15}+\ldots \end{aligned} \tag{7}$$

where $C'''_n \to (18)^{-1}(12)^{-3}n^6$ as $n \to \infty$. Clearly the asymptotic behaviour is dominated by loops of figure 2(4). The generating function corresponding to all loops is

$$\begin{aligned} G &= \sum_{i=1}^{4} G_i = \sum C_n x^n \\ &= x^6(1-x^2)^{-4}(1-x^3)^{-3}(1+x-2x^3-x^4+3x^5+4x^6+3x^7+x^{11}) \\ &= x^6+x^7+4x^8+5x^9+12x^{10}+17x^{11}+32x^{12}+48x^{13}+80x^{14}+119x^{15}+\ldots \end{aligned} \tag{8}$$

where $C_n \to n^6/31\,104$ as $n \to \infty$.

Finally we consider the general case. There are 68 categories of loops, including those shown in figures 1 and 2. The generating functions corresponding to figures 1 and 2 are the same as before except that, in equation (5), G_3 is replaced by

$$\sum_{a,b,c=1}^{\infty} (a+b)x^{3a+2b+2c}. \tag{9}$$

The overall asymptotic behaviour is still dominated by loops of figure 2(4). We find that the overall generating function is

$$\begin{aligned} G &= x^3(1-x^2)^{-4}(1-x^3)^{-3}(1+2x+x^2-x^3-x^4+6x^5+10x^6+4x^7 \\ &\qquad -6x^8-9x^9-2x^{10}+3x^{11}+3x^{12}-x^{14}) \\ &= x^3+2x^4+5x^5+10x^6+19x^7+37x^8+63x^9+109x^{10}+175x^{11} \\ &\qquad +277x^{12}+417x^{13}+623x^{14}+892x^{15}+\ldots \\ &= \sum C_n x^n \end{aligned} \tag{10}$$

where $C_n \to n^6/31\,104$ as $n \to \infty$. It is surprising that the exponent of n is different from the value for the square lattice.

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